REIDEMEISTER TORSION OF A 3-MANIFOLD OBTAINED BY A DEHN-SURGERY ALONG THE FIGURE-EIGHT KNOT

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Abstract. Let M be a 3-manifold obtained by a Dehn-surgery along the figure-eight knot. We give a formula of the Reisdemeisiter torsion of M for any $SL(2;\mathbb{C})$-irreducible representation. It has a rational expression of the trace of the image of the meridian.

1. Introduction

Reidemeister torsion is a piecewise linear invariant for manifolds and originally defined by Reidemeister, Franz and de Rham in 1930's. In 1980's Johnson developed a theory of the Reidemeister torsion from the view point of certain relation to the Casson invariant of a homology 3-sphere. He also derived an explicit formula for the Reidemeister torsion of a homology 3-sphere obtained by a $\frac{1}{n}$-Dehn surgery along any torus knot for $SL(2;\mathbb{C})$-irreducible representations. We generalized the Johnson's formula for any Seifert fibered space [2] along his studies.

In this paper, we give a formula for 3-manifolds obtained by Dehn surgeries along the figure-eight knot. Let $K \subset S^3$ be the figure-eight knot. The knot group $\pi_1(S^3 \setminus K)$ has the following presentation.

$$\pi_1(S^3 \setminus K) = \langle x, y \mid wx = yw \rangle$$

where $w = xy^{-1}x^{-1}y$, $l = w^{-1}\tilde{w}$ and $\tilde{w} = x^{-1}yxy^{-1}$. Now x is a meridian and l is a longitude.

Let M be a 3-manifold obtained by a $\dfrac{p}{q}$-surgery along K. The fundamental group $\pi_1(M)$ admits a presentation as follows;

$$\pi_1(M) = \langle x, y \mid wx = yw, x^p l^q = 1 \rangle.$$

Let $\rho : \pi_1(M) \to SL(2;\mathbb{C})$ be an irreducible representation. Assume the chain complex $C_*(M;\mathbb{C}^2_\rho)$ is acyclic. Then Reidemeister torsion $\tau_\rho(M) = \tau(C_*(M;\mathbb{C}^2_\rho))$ is given by the following.

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Theorem 1.1.
$$\tau_\rho(M) = \frac{2(u-1)}{u^2(u^2-5)}$$
where $u = tr(\rho(x))$.

Remark 1.2. We remark the trace u cannot move freely on the complex plane in the above formula. The value u depends on the surgery coefficient p, q.

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2. Definition of Reidemeister torsion

First let us describe the definition of the Reidemeister torsion for $SL(2;\mathbb{C})$-representations. Since we do not give details of definitions and known results, please see Johnson [1], Milnor [5, 6, 7] and Kitano [2, 3] for details.
Let W be an n-dimensional vector space over $\mathbb{C}$ and let $\mathbf{b} = (b_1, \cdots, b_n)$ and $\mathbf{c} = (c_1, \cdots, c_n)$ be two bases for W. Setting $b_i = \sum p_{ji}c_i$, we obtain a nonsingular matrix $P = (p_{ij})$ with entries in $\mathbb{C}$. Let $[\mathbf{b}/\mathbf{c}]$ denote the determinant of P.
Suppose
$$C_* : 0 \to C_m \overset{\partial_m}{\to} C_{m-1} \overset{\partial_{m-1}}{\to} \cdots \overset{\partial_2}{\to} C_1 \overset{\partial_1}{\to} C_0 \to 0$$
is an acyclic chain complex of finite dimensional vector spaces over $\mathbb{C}$. We assume that a preferred basis $\mathbf{c}_i$ for C_i is given for each i. Choose some basis $\mathbf{b}_i$ for $B_i = \mathrm{Im}(\partial_{i+1})$ and take a lift of it in C_{q+1}, which we denote by $\tilde{\mathbf{b}}_i$. Since $B_i = Z_i = \mathrm{Ker}\partial_i$, the basis $\mathbf{b}_i$ can serve as a basis for Z_i. Furthermore since the sequence
$$0 \to Z_i \to C_i \to B_{i-1} \to 0$$
is exact, the vectors $(\mathbf{b}_i, \tilde{\mathbf{b}}_{i-1})$ form a basis for C_i. Here $\tilde{\mathbf{b}}_{i-1}$ is a lift of $\mathbf{b}_{i-1}$ in C_i. It is easily shown that $[\mathbf{b}_i, \tilde{\mathbf{b}}_{i-1}/\mathbf{c}_i]$ does not depend on the choice of a lift $\tilde{\mathbf{b}}_{i-1}$. Hence we can simply denote it by $[\mathbf{b}_i, \mathbf{b}_{i-1}/\mathbf{c}_i]$.

Definition 2.1. The torsion $\tau(C_*)$ is given by the alternating product
$$\prod_{i=0}^{m} [\mathbf{b}_i, \mathbf{b}_{i-1}/\mathbf{c}_i]^{(-1)^{i+1}}.$$

Remark 2.2. It is easy to see that $\tau(C_*)$ does not depend on the choices of the bases $\{\mathbf{b}_0, \cdots, \mathbf{b}_m\}$.

Now we apply this torsion invariant of chain complexes to the following geometric situations. Let M be a finite CW-complex and $\tilde{M}$ a universal covering of M. The fundamental group $\pi_1(M)$ acts on $\tilde{M}$ as deck transformations. Then the chain complex $C_*(\tilde{M};\mathbb{Z})$ has the structure of a chain complex of free $\mathbb{Z}[\pi_1(M)]$-modules. We denote the 2-dimensional vector space $\mathbb{C}^2$ by V. Using a representation $\rho : \pi_1(M) \to SL(2;\mathbb{C})$, V has the structure of a $\mathbb{Z}[\pi_1(M)]$-module. Then we denote it by V_ρ and define the chain complex $C_*(M;V_\rho)$ by $C_*(\tilde{M};\mathbb{Z}) \otimes_{\mathbb{Z}[\pi_1(M)]} V_\rho$. Here we choose a preferred basis

$$\{\tilde{u}_1 \otimes \mathbf{e}_1, \tilde{u}_1 \otimes \mathbf{e}_2, \cdots, \tilde{u}_k \otimes \mathbf{e}_1, \tilde{u}_k \otimes \mathbf{e}_2\}$$

of $C_q(M;V_\rho)$ where $\{\mathbf{e}_1, \mathbf{e_2}\}$ is a canonical basis of $V = \mathbb{C}^2$ and $u_1, \cdots, u_k$ are the q-cells giving the preferred basis of $C_q(M;\mathbb{Z})$. We suppose that all homology groups $H_*(M;V_\rho)$ are vanishing. In this case we call ρ an acyclic representation.

Definition 2.3. Let $\rho : \pi_1(M) \to SL(2;\mathbb{C})$ be an acyclic representation. Then the Reidemeister torsion $\tau_\rho(M)$ is defined to be the torsion $\tau(C_*(M;V_\rho))$.

Remark 2.4.

(1) We define the $\tau_\rho(M) = 0$ for a non-acyclic representation ρ.
(2) The Reidemeister torsion $\tau_\rho(M)$ depends on several choices. However it is well known that the Reidemeister torsion is a piecewise linear invariant. See Johnson [1] and Milnor [5, 6, 7].

Here we recall the Reidemeister torsion of the torus and the solid torus.

Proposition 2.5. *Let $\rho : \pi_1(T^2) \to SL(2;\mathbb{C})$ be a representation.*

(1) *This representation ρ is an acyclic representation if and only if there exists an element $z \in \pi_1(T^2)$ such that* $\mathrm{tr}(\rho(z)) \neq 2$.
(2) *If ρ is acyclic, then it holds $\tau_\rho(T^2) = 1$.*

Next we consider the solid torus $S^1 \times D^2$ with $\pi_1(S^1 \times D^2) \cong \mathbb{Z}$ generated by x.

Proposition 2.6. *Let $\pi_1(S^1 \times D^2) \to SL(2;\mathbb{C})$ be a representation. Then it holds*

$$\begin{aligned}\tau(S^1 \times D^2; V_\rho) &= \frac{1}{\det(\rho(l) - E)} \\ &= \frac{1}{2 - \mathrm{tr}(\rho(l))}\end{aligned}$$

for a generator $l \in \pi_1(S^1 \times D^2) \cong \mathbb{Z}$. Here E is the identity matrix in $SL(2;\mathbb{C})$.

From here we assume M is a compact 3-manifold with an acyclic representation $\rho : \pi_1(M) \to SL(2;\mathbb{C})$. Here we take a torus decomposition of $M = A \cup_{T^2} B$. For simplicity, we write the same symbol ρ for a restricted representation to subgroups $\pi_1(A), \pi_1(B)$ and $\pi_1(T^2)$ of $\pi_1(M)$.
By this torus decomposition, we have the following exact sequence:

$$0 \to C_*(T^2; V_\rho) \to C_*(A; V_\rho) \oplus C_*(B; V_\rho) \to C_*(M; V_\rho) \to 0.$$

Proposition 2.7. *Let $\rho : \pi_1(M) \to SL(2;\mathbb{C})$ be a representation which restricted to $\pi_1(T^2)$ is acyclic. Then $H_*(M; V_\rho) = 0$ if and only if $H_*(A; V_\rho) = H_*(B; V_\rho) = 0$. In this case it holds*

$$\tau_\rho(M) = \tau_\rho(A)\tau_\rho(B).$$

We apply this proposition to any 3-manifold obtained by Dehn-surgery along a knot. Now let M be a closed 3-manifold obtained by a $\frac{p}{q}$-surgery along the figure eight knot K. Under the presentation

$$\pi_1(E(K)) = \langle x, y \mid wx = yw \rangle$$

where $w = xy^{-1}x^{-1}y$, $l = w^{-1}\tilde{w}$ and $\tilde{w} = x^{-1}yxy^{-1}$, x is a meridian and $l = w^{-1}\tilde{w}$ is a longitude.
We take an open tubular neighborhood $N(K)$ of K and its knot exterior $E(K) = S^3 \setminus N(K)$. We denote its closure of $N(K)$ by $\bar{N}$ which is homeomorphic to $S^1 \times D^2$. Since this 3-manifold M is obtained by Dehn-surgery along K, we have a torus decomposition

$$M = E(K) \cup \bar{N}.$$

Let $\rho : \pi_1(E(K)) = \pi_1(S^3 \setminus K) \to SL(2;\mathbb{C})$ be a representation which extends to $\pi_1(M)$. In this case it holds the following.

Proposition 2.8. *If ρ is acyclic on $\pi_1(T^2)$, then $\tau_\rho(M) = \tau_\rho(E(K))\tau_\rho(\bar{N})$. Further if all chain comeplexes are acyclic, then*

$$\tau_\rho(M) = \frac{\tau_\rho(E(K))}{2 - \mathrm{tr}(\rho(l))}.$$

3. Main result

Recall the following lemma, which is the fundamental way to study $SL(2;\mathbb{C})$-representations of a 2-bridge knot. Please see [8] as a reference.

Lemma 3.1. *Let $X, Y \in SL(2,\mathbb{C})$. If X and Y are conjugate and $XY \neq YX$, then there exists $P \in SL(2;\mathbb{C})$ s.t.*

$$PXP^{-1} = \begin{pmatrix} s & 1 \\ 0 & 1/s \end{pmatrix},\ PYP^{-1} = \begin{pmatrix} s & 0 \\ -t & 1/s \end{pmatrix}.$$

We apply this lemma to irreducible representations of $\pi_1(E(K))$. For any irreducible representation ρ, we may assume that its representative of this conjugacy class is given by

$$\rho_{s,t} : \pi_1(E(K)) \to SL(2;\mathbb{C})\ (s,t \in \mathbb{C} \setminus \{0\})$$

where

$$\rho_{s,t}(x) = \begin{pmatrix} s & 1 \\ 0 & 1/s \end{pmatrix}, \rho_{s,t}(y) = \begin{pmatrix} s & 0 \\ -t & 1/s \end{pmatrix}$$

Simply we write ρ to $\rho_{s,t}$ for some s, t. We compute the matrix

$$R = \rho(w)\rho(x) - \rho(y)\rho(w) = (R_{ij})$$

to get the defining equations of the space of the conjugacy classes of the irreducible representations.

- $R_{11} = 0$,
- $R_{12} = 3 - \frac{1}{s^2} - s^2 + 3t - \frac{t}{s^2} - s^2t + t^2$,
- $R_{21} = 3t - \frac{t}{s^2} - s^2t + 3t^2 - \frac{t^2}{s^2} - s^2t^2 - t^3 = tR_{12}$,
- $R_{22} = 0$.

Hence $R_{12} = 0$ is the equation of the space of the conjugacy classes of the irreducible representations.
This equation

$$3 - \frac{1}{s^2} - s^2 + 3t - \frac{t}{s^2} - s^2t + t^2 = 0$$

can be solved in t as

$$t = \frac{1 - 3s^2 + s^4 \pm \sqrt{1 - 2s^2 - s^4 - 2s^6 + s^8}}{2s^2}.$$

Here it can be seen that $L = \rho(l) = (l_{ij})$ is given by the followings:

Lemma 3.2.

$$l_{11} = 1 - \frac{t}{s^2} + s^2t - t^2 + \frac{t^2}{s^4} - \frac{t^2}{s^2} + s^2t^2 - t^3 - \frac{t^3}{s^2}$$

$$l_{12} = \frac{t}{s^3} + s^3t - \frac{t^2}{s} - st^2$$

$$l_{21} = \frac{t^2}{s^3} - \frac{2t^2}{s} - 2st^2 + s^3t^2 + \frac{t^3}{s^3} - \frac{2t^3}{s} - 2st^3 + s^3t^3 - \frac{t^4}{s} - st^4$$

$$l_{22} = 1 + \frac{t}{s^2} - s^2t - t^2 + \frac{t^2}{s^2} - s^2t^2 + s^4t^2 - t^3 - s^2t^3$$

Here we get the trace of direct computation.

$$\mathrm{tr}(\rho(l)) = 2 - 2t^2 + \frac{t^2}{s^4} + s^4t^2 - 2t^3 - \frac{t^3}{s^2} - s^2t^3$$

It is easy to see that $\mathrm{tr}(\rho(l)) \neq 2$ if $u = s + \frac{1}{s} = 2$. Hence there exists an element $z \in \pi_1(T^2)$ s.t. $\mathrm{tr}(\rho(z) \neq 2$. This means ρ is always acyclic on T^2. Now we have

$$\tau_\rho(M) = \tau_\rho(E(K))\tau_\rho(\bar{N}).$$

Here we obtain the Reidemeister torsion of $E(K)$ as follows. See [3] for precise computation.

Proposition 3.3.

$$\tau_\rho(E(K)) = -2(u-1)$$

where $u = s + \frac{1}{s}$.

By substituting

$$t = \frac{1 - 3s^2 + s^4 \pm \sqrt{1 - 2s^2 - s^4 - 2s^6 + s^8}}{2s^2}$$

in $\mathrm{tr}(\rho(l)$, we get the following proposition.

Proposition 3.4.

$$\tau_\rho(\bar{N}) = -\frac{1}{u^2(u^2-5)}.$$

Therefore we obtain the following formula:

$$\begin{aligned}\tau_\rho(M) &= \tau_\rho(E(K))\tau_\rho(\bar{N}) \\ &= (-2(u-1))\left(-\frac{1}{u^2(u^2-5)}\right) \\ &= \frac{2(u-1)}{u^2(u^2-5)}.\end{aligned}$$

Remark 3.5. The representations for $u^2 - 5 = 0$ are degenerate into reducible representation from irreducible representations.

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